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Hierarchical Mass Matrices in a Minimal $SO(10)$ Grand Unification II

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Abstract

We continue to investigate the minimal $SO(10) \times U(1)_H$ model which we proposed recently. Renormalization group analysis of the model results in natural predictions of quark-lepton masses and Kobayashi-Maskawa matrix along with neutrino mixings adequate for solar neutrino oscillation.

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1 Introduction

Grand unification has been a fascinating idea for two decades. Among the possible gauge groups, $SO(10)$ is the smallest candidate that can incorporate the observed fermions of one generation into an irreducible multiplet: it attains matter unification for a single generation of quarks and leptons. However, $SO(10)$ grand unification by itself provides no natural place for triplicity of generations, to say nothing of the hierarchical structure of mass matrices. For instance, it gives no explanation on the fact that the mass of top quark is more than 10^6 times that of electron.

In a previous paper [1], we constructed an $SO(10) \times U(1)_H$ model as an attempt to place generation structure on a plausible position in grand unification. We introduced a *minimal* Higgs content to break the gauge symmetry without any additional scalars. The horizontal Peccei-Quinn symmetry $U(1)_H$ distinguishes three generations of fermions to impose Georgi-Jarlskog(GJ) relations [2] at the unification scale M_U. It also makes Yukawa couplings of order one be able to realize hierarchical mass matrices with the aid of remnant effects of certain irrelevant terms suppressed by the Planck scale M_P as the cut-off scale in the theory (For more details, see ref.[1]).

The model has three factors which determine the values of mass matrices in combination with order-one Yukawa couplings at the unification scale. The first one is the hierarchy factor $\varepsilon = M_U/M_P$ brought about by the remnant effects mentioned above. The second one comes from the mixing of Higgs bosons which is represented by parameters α, β, γ, δ defined later. The third is the running of Yukawa couplings according to renormalization group (RG) equations.

In this paper, we proceed to analyze mainly the third factor to compare predictions of our model with experimental estimates. Numerical analysis of RG equations results in natural predictions of quark-lepton masses and Kobayashi-Maskawa(KM) matrix along with neutrino mixings adequate for solar neutrino oscillation [3].

The paper is organized as follows: We first recapitulate the setup of our model in section 2 to derive RG equations in section 3. We restrict ourselves to dealing with one-loop RG equations and ignore threshold corrections throughout the paper. Section 4 makes exposition of numerical results obtained by RG analysis. Section 5 concludes the paper. Some definitions are given in Appendix A and analytical consideration on quark mass matrices is made in Appendix B.

2 Yukawa Interactions

In this section, we present effective Yukawa interactions under the chain of symmetry breaking considered in ref.[1]. There are three scales of symmetry breaking postulated (with the help of fine-tuning) in the model: the unification scale M_U, the intermediate scale M_I, and the weak scale M_W.

We introduced Higgs fields $\Phi(210,-8)$, $\bar{\Delta}(\overline{126},-10)$, and $H(10,-2)$ transforming under $SO(10)\times U(1)_H$. The field $\Phi(210,-8)$ develops a VEV of order M_U, which breaks $SO(10)\times U(1)_H$ into $G_{224}\times Z_8$, where G_{224} denotes the Pati-Salam group $SU(2)_L\times SU(2)_R\times SU(4)_C$. In particular, D-parity oddness of $\Phi(210,-8)$ causes violation of the left-right symmetry.

We assumed that there remain $H(2,2,1)$, $\bar{\Delta}(2,2,15)$, and $\bar{\Delta}(1,3,\overline{10})$ on the Pati-Salam stage between the scales M_U and M_I, the other Higgs fields with masses of order M_U decoupling from the system. This is a modification of the minimal fine-tuning, which claims non-decoupling of only $H(2,2,1)$ and $\bar{\Delta}(1,3,\overline{10})$. The additional field $\bar{\Delta}(2,2,15)$ develops an induced VEV [4] of a considerable size to contribute fermion mass matrices [1].

The effective Yukawa interactions on this stage are given by

$$\begin{aligned}\mathcal{L}_1 &= Y_1\psi_L^T\phi_1\psi_R + Y_2\psi_L^T\widetilde{\phi}_1\psi_R \\ &\quad + Y_3\psi_L^T\phi_{15}\psi_R + Y_4\psi_L^T\widetilde{\phi}_{15}\psi_R + Y_5\psi_R^T\phi_{10}\psi_R + h.c.,\end{aligned} \tag{2.1}$$

where Y's are Yukawa couplings and summation over suppressed generation indices should be understood. Fermions ψ_L and ψ_R denote $(2,1,4)$ and $(1,2,\overline{4})$ representations, respectively. Scalars ϕ_1, ϕ_{15}, and ϕ_{10} correspond to $H(2,2,1)$, $\bar{\Delta}(2,2,15)$, and $\bar{\Delta}(1,3,\overline{10})$ in this turn (See Appendix A).

Yukawa couplings at the unification scale M_U provide a boundary condition

$$Y_1 = \begin{pmatrix} -\varepsilon^2 y_{11} & 0 & \varepsilon y_{13} \\ 0 & 0 & 0 \\ -\varepsilon y_{13} & 0 & y_{33}\end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & \varepsilon^2 y_{12} & 0 \\ \varepsilon^2 y_{12} & 0 & -\varepsilon y_{23} \\ 0 & \varepsilon y_{23} & 0\end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & 0 & z_{13} \\ 0 & z_{22} & 0 \\ z_{13} & 0 & 0\end{pmatrix}, \qquad Y_4 = 0, \qquad Y_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon z_{11} & 0 & z_{13} \\ 0 & z_{22} & 0 \\ z_{13} & 0 & \varepsilon z_{33}\end{pmatrix}, \tag{2.2}$$

where Y's have been regarded as matrices in generation space. The small factor $\varepsilon = M_U/M_P$ stems from the remnant effects and y's and z's are input parameters of order one, which we expect to be determined by more fundamental theory presumably including gravitation. GJ relations can be obtained from the above texture as shown in ref.[1] (See also Appendix B).

Below the intermediate scale M_I, the model effectively coincides with the standard model, and the effective Yukawa interactions are given by

$$\mathcal{L}_2 \;=\; Y_d^\dagger\,(\overline{Q}\,\phi)\,d_R + Y_u^\dagger\,(\overline{Q}\,\widetilde{\phi})\,u_R + Y_e^\dagger\,(\overline{L}\,\phi)\,e_R + Y_\nu^\dagger\,(\overline{L}\,\widetilde{\phi})\,\nu_R + h.c., \tag{2.3}$$

where ϕ, Q, and L denote the standard Higgs, quark, and lepton doublets, respectively. Although this Lagrangian contains Dirac mass terms for neutrinos $Y_\nu^\dagger(\overline{L}\widetilde{\phi})\nu_R$, they can be approximately neglected [5] below M_I due to order-M_I Majorana masses of right-handed neutrinos ν_R, which originates from the term $Y_5\psi_R^T\phi_{10}\psi_R$ in eq.(2.1).

The standard Higgs doublet ϕ is a linear combination [6] of four doublets contained in $H(2,2,1)$ and $\bar{\Delta}(2,2,15)$:

$$\phi = \alpha H_{\frac{1}{2}} + \beta \widetilde{H}_{-\frac{1}{2}} + \gamma\bar{\Delta}_{\frac{1}{2}} + \delta\widetilde{\bar{\Delta}}_{-\frac{1}{2}} \tag{2.4}$$

with a normalization condition

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1, \tag{2.5}$$

where subscripts $\pm\frac{1}{2}$ indicate hypercharges of the fields. Hence the following relations hold at the intermediate scale M_I:

$$\begin{aligned}
Y_u &= \alpha Y_1 + \beta Y_2 + \frac{1}{2\sqrt{3}}\,\gamma Y_3 + \frac{1}{2\sqrt{3}}\,\delta Y_4, \\
Y_d &= \alpha Y_2 + \beta Y_1 + \frac{1}{2\sqrt{3}}\,\gamma Y_4 + \frac{1}{2\sqrt{3}}\,\delta Y_3, \\
Y_\nu &= \alpha Y_1 + \beta Y_2 + \frac{-3}{2\sqrt{3}}\,\gamma Y_3 + \frac{-3}{2\sqrt{3}}\,\delta Y_4, \\
Y_e &= \alpha Y_2 + \beta Y_1 + \frac{-3}{2\sqrt{3}}\,\gamma Y_4 + \frac{-3}{2\sqrt{3}}\,\delta Y_3,
\end{aligned} \tag{2.6}$$

where the coefficients $\frac{1}{2\sqrt{3}}$ and $\frac{-3}{2\sqrt{3}}$ result from the normalization of ϕ_{15} defined in the Appendix A. The parameters α, β, γ, δ are written in terms of the VEVs defined in ref.[1] as follows:

$$\alpha = \frac{v_t}{v_W}, \quad \beta = \frac{v_b}{v_W}, \quad \gamma = 2\sqrt{3}\frac{w_c^*}{v_W}, \quad \delta = 2\sqrt{3}\frac{w_s^*}{v_W}, \tag{2.7}$$

where v_W denote the VEV of the standard Higgs doublet. Note that the VEVs w_c^* and w_s^* are of order ε relative to v_W under the assumption in ref.[1] that they are induced in the Higgs ϕ_{15} with mass of order M_I. Hence γ and δ take values of order ε. This makes it natural to define order-one quantity γ' and δ' in terms of

$$\gamma = \gamma'\varepsilon, \qquad \delta = \delta'\varepsilon. \tag{2.8}$$

It should be emphasized that only two parameters ε and β are small to realize the hierarchical structure of mass matrices (See Appendix B). The other parameters y's, z's, γ', and δ' are of order one, which merely affect detailed numerical values of masses and mixings without altering their orders of magnitude.

3 Renormalization Group Equations

In this section, we show one-loop RG equations for gauge and Yukawa [7] couplings (g's and Y's) in the effective theories described in the previous section.

We first give RG equations for the G_{224} theory on the stage between the unification and intermediate scales:

$$\frac{d\omega_i}{dt} = -\frac{1}{2\pi}b_i; \qquad b_i = \left(2, \frac{26}{3}, -\frac{7}{3}\right), \qquad i = 2_L,\ 2_R,\ 4_C, \tag{3.1}$$

where $\omega_i = \frac{4\pi}{g_i^2}$, $t = \ln\mu$, and μ denotes a renormalization point in the $\overline{\mathrm{MS}}$ scheme; and

$$\begin{aligned}
16\pi^2\frac{dY_1}{dt} &= Y_1\beta_L + \beta_R^\dagger Y_1 + Y_1\beta_1 + \beta_{v1} + Y_1\beta_g, \\
16\pi^2\frac{dY_2}{dt} &= Y_2\beta_L + \beta_R^\dagger Y_2 + Y_2\beta_1 + \beta_{v2} + Y_2\beta_g, \\
16\pi^2\frac{dY_3}{dt} &= Y_3\beta_L + \beta_R^\dagger Y_3 + Y_3\beta_{15} + \beta_{v3} + Y_3\beta_g, \\
16\pi^2\frac{dY_4}{dt} &= Y_4\beta_L + \beta_R^\dagger Y_4 + Y_4\beta_{15} + \beta_{v4} + Y_4\beta_g, \\
16\pi^2\frac{dY_5}{dt} &= Y_5\beta_R + \beta_R^\dagger Y_5 + Y_5\beta_{10} + Y_5\beta'_g,
\end{aligned} \tag{3.2}$$

where β_L, β_R and $\beta_{1,\ 15,\ 10}$ correspond to contributions from wave-function renormalization of ψ_L, ψ_R, and $\phi_{1,\ 15,\ 10}$, respectively; β_v's and β_g, β'_g correspond to contributions from vertex renormalization and gauge couplings:

$$\begin{aligned}
\beta_L &= Y_1^\dagger Y_1 + Y_2^\dagger Y_2 + \frac{15}{4}(Y_3^\dagger Y_3 + Y_4^\dagger Y_4), \\
\beta_R &= Y_1^\dagger Y_1 + Y_2^\dagger Y_2 + \frac{15}{4}(Y_3^\dagger Y_3 + Y_4^\dagger Y_4 + Y_5^\dagger Y_5), \\
\beta_1 &= 4\ \mathrm{tr}(Y_1^\dagger Y_1 + Y_2^\dagger Y_2), \\
\beta_{15} &= \mathrm{tr}(Y_3^\dagger Y_3 + Y_4^\dagger Y_4), \\
\beta_{10} &= \mathrm{tr}(Y_5^\dagger Y_5), \\
\beta_{v1} &= -2Y_1Y_2^\dagger Y_2 - 2Y_2Y_2^\dagger Y_1 - \frac{15}{2}Y_3Y_2^\dagger Y_4 - \frac{15}{2}Y_4Y_2^\dagger Y_3,
\end{aligned}$$

$$
\begin{aligned}
\beta_{v2} &= -2Y_2Y_1^\dagger Y_1 - 2Y_1Y_1^\dagger Y_2 - \frac{15}{2}Y_3Y_1^\dagger Y_4 - \frac{15}{2}Y_4Y_1^\dagger Y_3, \\
\beta_{v3} &= \frac{1}{2}Y_3Y_4^\dagger Y_4 + \frac{1}{2}Y_4Y_4^\dagger Y_3 - 2Y_1Y_4^\dagger Y_2 - 2Y_2Y_4^\dagger Y_1, \\
\beta_{v4} &= \frac{1}{2}Y_3Y_3^\dagger Y_4 + \frac{1}{2}Y_4Y_3^\dagger Y_3 - 2Y_1Y_3^\dagger Y_2 - 2Y_2Y_3^\dagger Y_1, \\
\beta_g &= \frac{9}{4}g_L^2 + \frac{9}{4}g_R^2 + \frac{15}{4}g_{4C}^2, \\
\beta_g' &= \frac{9}{2}g_R^2 + \frac{15}{4}g_{4C}^2. \qquad (3.3)
\end{aligned}
$$

Let us turn to the energy region below the intermediate scale. RG equations on this stage are those for the standard model:

$$
\frac{d\omega_i}{dt} = -\frac{1}{2\pi}b_i; \qquad b_i = \left(\frac{41}{10}, -\frac{19}{6}, -7\right), \qquad i = 1_Y,\ 2_L,\ 3_C; \qquad (3.4)
$$

$$
\begin{aligned}
16\pi^2\frac{dY_u}{dt} &= Y_u\left[\frac{3}{2}(Y_u^\dagger Y_u - Y_d^\dagger Y_d) + \mathrm{tr}(3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e) - \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\right], \\
16\pi^2\frac{dY_d}{dt} &= Y_d\left[\frac{3}{2}(Y_d^\dagger Y_d - Y_u^\dagger Y_u) + \mathrm{tr}(3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e) - \left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\right], \\
16\pi^2\frac{dY_e}{dt} &= Y_e\left[\frac{3}{2}Y_e^\dagger Y_e + \mathrm{tr}(3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e) - \frac{9}{4}(g_1^2 + g_2^2)\right]. \qquad (3.5)
\end{aligned}
$$

For a recent analysis of them, see Ref.[8].

4 Numerical Results

We proceed to consider numerical solutions to the RG equations listed in the previous section. The running of the gauge couplings are independent of Yukawa couplings to the extent of one-loop analysis. Thus we can first obtain the unification scale M_U and the intermediate scale M_I [9] by means of RG equations for gauge couplings (3.1) and (3.4) with their experimental values below the weak scale [10] as a boundary condition:

$$
M_U \simeq 10^{16.7}\,\mathrm{GeV}, \quad M_I \simeq 10^{11.2}\,\mathrm{GeV}. \qquad (4.1)
$$

The unified gauge coupling at M_U comes out to be $g_U \simeq 0.585$. We think of this as a typical value of order one, since gauge coupling seems fundamental in view of its geometrical origin. Yukawa couplings at the unification scale are to be compared with this value as the standard one. (Conversely, one might also regard this value as an experimental evidence of coupling constants being of order one.)

Now that we have obtained running gauge couplings, we turn to analyzing Yukawa sector. We assume CP conservation in the following analysis. In particular, KM phase is set to zero. Thus the case of large KM phase are excluded from the analysis in this paper, though small phase may be taken into account perturbatively and seems not to affect the results considerably.

The procedure we pursue is as follows: To begin with, we choose appropriate values for the input parameters ε, β, y's, z's, γ', and δ' partly with the help of trial and error (See Appendix B). Note that y's, z's, γ', and δ' must be of order one. Then we make Yukawa couplings evolve from the unification scale M_U down to the weak scale $M_W \simeq 174\text{GeV}$ by solving RG equations numerically. Finally the resultant Yukawa matrices at the weak scale are diagonalized to yield fermion masses and mixings. Neutrino masses and mixings are derived by means of sea-saw approximation [3] from the values of Yukawa couplings at the intermediate scale M_I, where the right-handed neutrinos are supposed to decouple. We compare the results with experimental estimates of running masses and mixings at the weak scale [11].

Let us exhibit a numerical sample which provides a realistic pattern of mass matrices at the weak scale: input parameters in table 1 lead to the results in tables 2 and 3. The masses of neutrinos are given by

$$(\mathrm{m}_{\nu_e},\ \mathrm{m}_{\nu_\mu},\ \mathrm{m}_{\nu_\tau}) \sim (4.1\times 10^{-11},\ 5.9\times 10^{-5},\ 3.0\times 10^{-2})\times \frac{v_W^2}{v_I}, \tag{4.2}$$

where v_I denotes the VEV developed by the Higgs ϕ_{10}. The values $v_W \simeq 174\text{GeV}$ and $v_I \sim 10^{11.2}\text{GeV}$ predict

$$m_{\nu_\mu} \sim 10^{-3}\text{eV}, \tag{4.3}$$

and a negligible value of m_{ν_e} relative to m_{ν_μ}. This is consistent with the small-angle MSW solution to the solar neutrino problem [3], which is implemented by

$$\begin{aligned} \Delta m_{e\mu} &= (m_{\nu_\mu}^2 - m_{\nu_e}^2)^{1/2} \sim 10^{-3}\text{eV}, \\ \sin\theta_{e\mu} &\simeq 0.03 - 0.06. \end{aligned} \tag{4.4}$$

The above example shows that qualitative agreement is achieved between predictions of the model and experimental estimates. In particular, the hierarchical structure of mass matrices was shown to be indeed realized in terms of Yukawa couplings exclusively of order one.

So far so good. However, we cannot help complaining about the prediction of bottom mass. It has tendency to come out larger [12] in this model than the experimental estimate (provided the tau mass is fitted.) This defect originates from GJ relation $m_b \simeq m_\tau$ at the unification

$$Y_1 = \begin{pmatrix} 0.3\times\varepsilon^2 & 0 & 0.3\times\varepsilon \\ 0 & 0 & 0 \\ -0.3\times\varepsilon & 0 & 0.5 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 3.25\times\varepsilon^2 & 0 \\ 3.25\times\varepsilon^2 & 0 & 0.16\times\varepsilon \\ 0 & -0.16\times\varepsilon & 0 \end{pmatrix},$$

$$Y_5 = \begin{pmatrix} 0.5\times\varepsilon & 0 & -0.145 \\ 0 & 0.65 & 0 \\ -0.145 & 0 & 0.5\times\varepsilon \end{pmatrix},$$

$$\varepsilon = \frac{1}{250}, \qquad \beta = \frac{1}{53}, \qquad \gamma' = \frac{25}{9}, \qquad \delta' = \frac{1}{3}.$$

Table 1: Sample input parameters at $\mu = 10^{16.7}$GeV.

	Prediction	Experiment		Prediction	Experiment
m_u	2.5×10^{-3}	2.4×10^{-3}	m_d	3.3×10^{-3}	4.2×10^{-3}
m_c	0.61	0.61	m_s	0.087	0.085
m_t	160		m_b	3.7	2.9
	Prediction	Experiment		Prediction	Experiment
m_e	5×10^{-4}	5×10^{-4}	m_{ν_e}	$\sim 4\times10^{-19}$	
m_μ	0.1	0.1	m_{ν_μ}	$\sim 6\times10^{-13}$	
m_τ	1.7	1.7	m_{ν_τ}	$\sim 3\times10^{-10}$	

Table 2: Running quark and lepton masses (GeV) at $\mu = 174$GeV.

$$\begin{aligned} V_{\text{quark}} &= \begin{pmatrix} -0.98 & 0.22 & -0.0063 \\ -0.22 & -0.98 & 0.052 \\ -0.005 & -0.052 & -1 \end{pmatrix} \\ V_{\text{lepton}} &= \begin{pmatrix} 1 & -0.058 & -0.017 \\ 0.059 & 1 & 0.062 \\ 0.014 & -0.063 & 1 \end{pmatrix} \end{aligned}$$

Table 3: Predictions of quark and lepton mixing matrices at $\mu = 174$GeV.

scale and is scarcely dependent on the parameters we can choose to get realistic predictions. Taking this issue seriously, we might need some modifications of the present model. Even in such circumstances, we hope that the qualitative features of the model survive to naturally achieve the mass-matrix hierarchy.

5 Concluding Remarks

In the preceding sections, we have investigated the running of Yukawa couplings below the unification scale in our model of grand unification [1]. The remnant effects of orders ε and ε^2 play crucial roles to make realistic predictions in the model. Without the remnants, the model would possess a parity symmetry for the second generation $\psi_2 \to -\psi_2$, which completely forbids mixings between the second and the other generations: that is, in the renormalizable setting, radiative corrections could not generate necessary operators corresponding to the remnants.

Finally let us see the behavior of couplings above the unification scale. The Yukawa interactions are given by

$$\mathcal{L}_0 = \frac{1}{4}\, Y_{10} \psi^T B \gamma_\mu H_\mu \psi + \frac{1}{4 \cdot 5!}\, Y_{126} \psi^T B \gamma_{\mu_1 \mu_2 \cdots \mu_5} \bar{\Delta}_{\mu_1 \mu_2 \cdots \mu_5} \psi + h.c., \tag{5.1}$$

where higher-dimensional terms are ignored. Greek indices are $SO(10)$ vector ones, γ_μ yields 32-dimensional representation of Clifford algebra $\{\, \gamma_\mu \,,\, \gamma_\nu \,\} = 2\, \delta_{\mu\nu}$, and B denotes charge conjugation for $SO(10)$ spinors: $B = \prod_{\mu:\,\mathrm{odd}} \gamma_\mu$. 126 Higgs $\bar{\Delta}$ is self-dual antisymmetric tensor, which satisfy

$$\bar{\Delta}_{\mu_1 \mu_2 \cdots \mu_5} = \frac{i}{5!}\, \epsilon_{\mu_1 \mu_2 \cdots \mu_5 \mu_6 \mu_7 \cdots \mu_{10}}\, \bar{\Delta}_{\mu_6 \mu_7 \cdots \mu_{10}}\,, \tag{5.2}$$

where $\epsilon_{\mu_1 \mu_2 \cdots \mu_{10}}$ denotes the invariant antisymmetric tensor.

The one-loop RG equation for the unified gauge coupling g is given by

$$\frac{d\omega}{dt} = -\frac{1}{2\pi} \frac{16}{3}, \tag{5.3}$$

which shows that the theory is asymptotically non-free. Note that the absolute value of the beta function is so small that the coupling stays within perturbative range up to the Planck scale. The running of effective gauge couplings are exhibited in figure 1.

The Yukawa couplings above the unification scale obey approximate RG equations

$$\begin{aligned} 16\pi^2 \frac{dY_{10}}{dt} \;&=\; \frac{10}{16}\, Y_{10} Y_{10}^\dagger Y_{10} + \frac{63}{8}\, Y_{126} Y_{126}^\dagger Y_{10} + \frac{63}{8}\, Y_{10} Y_{126}^\dagger Y_{126} \\ &\quad + Y_{10} \mathrm{tr}(Y_{10} Y_{10}^\dagger) - 24\, g^2 Y_{10}, \end{aligned} \tag{5.4}$$

$$16\pi^2 \frac{dY_{126}}{dt} = \frac{63}{4} Y_{126} Y_{126}^\dagger Y_{126} + \frac{5}{16} Y_{10} Y_{10}^\dagger Y_{126} + \frac{5}{16} Y_{126} Y_{10}^\dagger Y_{10} + 2\, Y_{126} \mathrm{tr}(Y_{126} Y_{126}^\dagger) - 24\, g^2 Y_{10} \tag{5.5}$$

At the unification scale, they satisfy

$$(Y_{10})_{33} = (Y_1)_{33}, \qquad 2\, Y_{126} = Y_3. \tag{5.6}$$

The flow of Yukawa coupling $(Y_{10})_{33}$ with respect to the unified gauge coupling g is shown in figure 2. The Plank scale corresponds to $g \simeq 0.62$ and the GUT scale to $g \simeq 0.585$. Thus the Yukawa coupling y_{33} in the numerical sample in the previous section is found to be of order one even at the Planck scale. This is consistent with the perturbative treatment above and the general philosophy of effective field theory that coupling constants are of order one at the cut-off scale.

In fact, figure 2 suggests that $(Y_{10})_{33}$ is almost always of order one at the GUT scale whatever it is at the Plank scale. One can even consider the case where the Yukawa coupling blows up at the Plank scale, which might indicate the presence of some dynamical phenomenon out of perturbative picture.

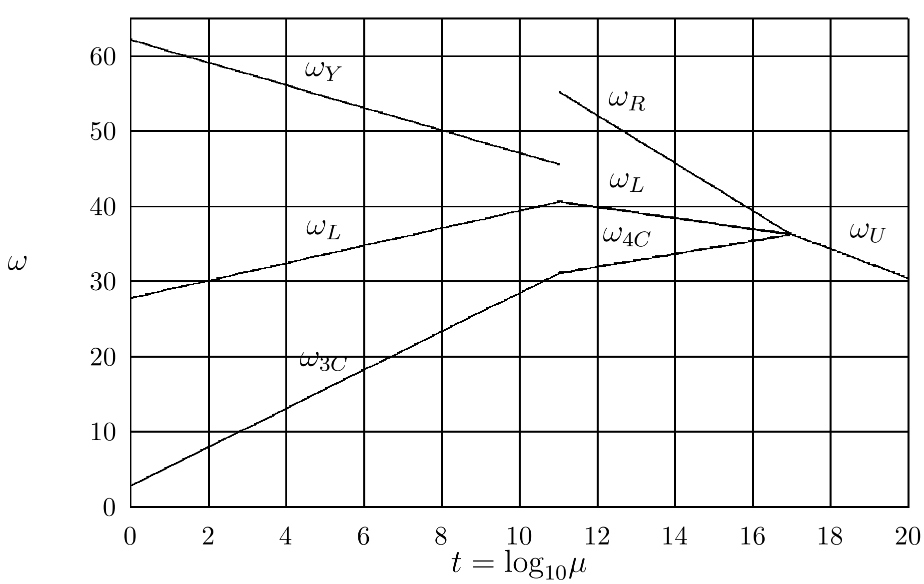

Figure 1: Running gauge couplings;
$\omega = 4\pi/g^2$ and μ is a renormalization point in GeV unit.

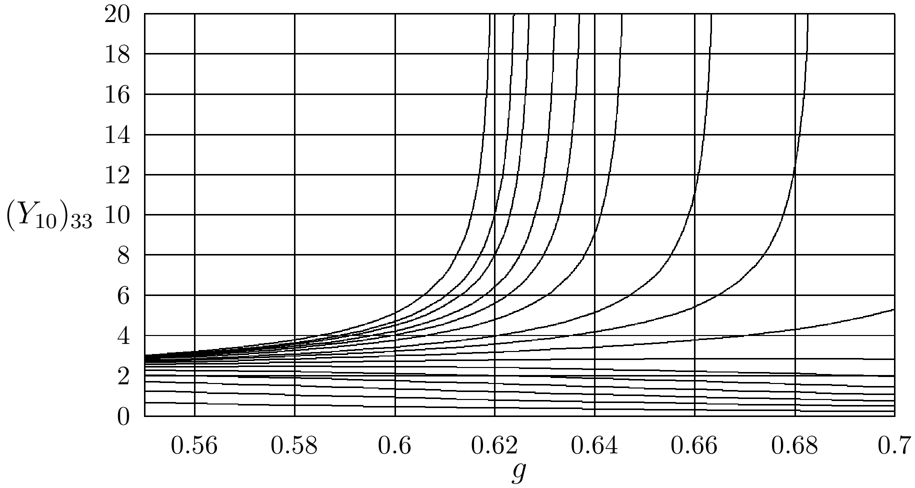

Figure 2: Running Yukawa coupling above M_U;
g denotes the unified gauge coupling.

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Appendix

A Definitions

On the G_{224} stage, the following representations of the fields under $SU(2)_R \times SU(2)_L$ are employed:

$$\psi_R^T = (U_R\ D_R), \qquad \psi_L^T = (U_L\ D_L),$$

$$\phi_{1,15} = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}_{1,\ 15}, \qquad \phi_{10} = \phi_{10}^m \tau^m; \quad \tau^m = \frac{1}{\sqrt{2}}\sigma^m, \quad m = 1,2,3 \tag{A.1}$$

where U and D represent quartets of $SU(4)_C$ and σ^m denote Pauli matrices. The definition of $\widetilde{\phi}_{1,15}$ is given by

$$\widetilde{\phi}_{1,15} = \epsilon^T \phi_{1,15}^* \epsilon; \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

The $SU(4)_C$ representations of ϕ_{15} and ϕ_{10} are given by

$$\phi_{15} = \sum_{a=1}^{15} \phi_{15}^a T^a; \quad \mathrm{tr}(T^a T^b) = \delta^{ab}, \quad \sum_{a=1}^{15} T^a T^a = \frac{15}{4}\mathbf{1}, \quad \sum_{b=1}^{15} T^b T^a T^b = -\frac{1}{4}T^a,$$

$$\phi_{10} = \sum_{\alpha=1}^{10} \phi_{10}^\alpha S_\alpha; \quad \mathrm{tr}(S_\alpha S^\beta) = \delta_\alpha^\beta, \quad \sum_{\alpha=1}^{10} S_\alpha S^\alpha = \frac{5}{2}\mathbf{1}, \quad S^\alpha = S_\alpha^*, \tag{A.3}$$

where T^a correspond to $SU(4)_C$ generators in the defining representation.

B Quark Mass Matrices

It seems instructive to consider some analytical relations among quark masses and mixings which are expected to hold at the weak scale M_W in our model. We make a rough estimate that the running of each Yukawa coupling between the scales M_U and M_W does not affect its order of magnitude, which can be checked numerically. Then the mass matrices at the weak scale can be written in terms of rescaled couplings

$$y_{33}^u \sim y_{33}^d \sim \alpha y_{33}, \quad z_{22}^u \sim z_{22}^d \sim \frac{\gamma'}{2\sqrt{3}} z_{22}, \quad etc. \tag{B.1}$$

as follows:

$$
\begin{aligned}
M_u &= \begin{pmatrix} -\varepsilon^2 y^u_{11} & \varepsilon^2\ y^u_{12} & \varepsilon(z^u_{13}+y^u_{13}) \\ \varepsilon^2\xi y^u_{12} & \varepsilon z^u_{22} & -\varepsilon\xi y^u_{23} \\ \varepsilon(z^u_{13}-y^u_{13}) & \varepsilon\xi y^u_{23} & y^u_{33} \end{pmatrix} v_W, \\
M_d &= \begin{pmatrix} -\varepsilon^2 y^d_{11} & \varepsilon^2\xi^{-1} y^d_{12} & \varepsilon(\xi^{-1}\zeta z^d_{13}+y^d_{13}) \\ \varepsilon^2\xi^{-1} y^d_{12} & \varepsilon\xi^{-1}\zeta z^d_{22} & -\varepsilon\xi^{-1} y^d_{23} \\ \varepsilon(\xi^{-1}\zeta z^d_{13}-y^d_{13}) & \varepsilon\xi^{-1} y^d_{23} & y^d_{33} \end{pmatrix} \xi v_W,
\end{aligned} \tag{B.2}
$$

where

$$
\xi = \frac{\beta}{\alpha}, \quad \zeta = \frac{\delta'}{\gamma'}. \tag{B.3}
$$

Eq.(3.5) implies that the ratio between the Yukawa couplings of top and bottom quarks does not change considerably

$$
y^u_{33} \simeq y^d_{33} \tag{B.4}
$$

under dominance of the gauge couplings g_2 and g_3.

We now derive approximate relations for the parameters ξ and ζ with the aid of smallness of the hierarchy factor $\varepsilon = M_U/M_P \simeq 1/250$ (See section 3). The masses of top and bottom quarks are expressed as

$$
\frac{m_t}{v_W} = y^u_{33} + \mathrm{O}(\varepsilon^2), \quad \frac{m_b}{\xi v_W} = y^d_{33} + \mathrm{O}(\varepsilon^2\xi^{-2}), \tag{B.5}
$$

which implies

$$
\xi \simeq \frac{m_b}{m_t}. \tag{B.6}
$$

If we put $m_t \simeq 160\mathrm{GeV}$, we have $\xi \simeq 1/50$, which yields $\varepsilon\xi^{-1} \simeq 1/5$.

Let us proceed to the second generation. We obtain

$$
\frac{m_c}{v_W} = \varepsilon z^u_{22} + \varepsilon^2\xi\frac{{y^u_{23}}^2}{y^u_{33}} + \mathrm{O}(\varepsilon^3), \quad \frac{m_s}{\xi v_W} = \varepsilon\xi^{-1}\zeta z^d_{22} + \varepsilon^2\xi^{-2}\frac{{y^d_{23}}^2}{y^d_{33}} + \mathrm{O}(\varepsilon^3\xi^{-3}) \tag{B.7}
$$

with a KM matrix element

$$
V_{cb} = \varepsilon(\xi^{-1}\frac{y^d_{23}}{y^d_{33}} - \xi\frac{y^u_{23}}{y^u_{33}}) + \mathrm{O}(\varepsilon^2\xi^{-2}). \tag{B.8}
$$

Note that the $\mathrm{O}(\epsilon^2\xi^{-2})$ term in eq.(B.7) is retained in anticipation of smallness of ζ. By means of these relations, we get

$$
\zeta \simeq \frac{1}{m_c}(m_s - V_{cb}^2 m_b), \tag{B.9}
$$

which is satisfied when $\zeta \simeq 1/10$. The enhancement factor ξ^{-1} for the mixing V_{cb} in eq.(B.8) comes from contribution of $\widetilde{H}$, whose coupling is characteristic of non-supersymmetric models. A choice $y^d_{23}/y^d_{33} \simeq 1/4$ yields $V_{cb} \simeq 1/20$.

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